A Piecewise Decomposition of Cardinality Constrained Optimization Problems

Master Thesis under the Supervision of Alexandra Schwartz

Topic
Consider an optimization problem with cardinality constraints
\[
\min f(x) \quad \text{s.t.} \quad x \in X, \quad |\text{supp}(x)| \leq \kappa.
\]
This type of problem appears e.g. in portfolio optimization, where one wants to identify at most \( \kappa \) optimal investment opportunities \( i \) and the corresponding optimal investment \( x_i \). However, these optimization problems are theoretically and numerically challenging since it the noncontinuous constraint function \( |\text{supp}(x)| \).

Given a feasible point \( x^* \) of the cardinality constrained problem define
\[
I_0(x^*) := \{ i | x_i = 0 \}.
\]
Then the cardinality constrained problem can locally be decomposed into the pieces \( \text{NLP}(I) \)
\[
\min_x f(x) \quad \text{s.t.} \quad x \in X, \quad x_i = 0 \quad \forall i \in I,
\]
for all subsets \( I \subseteq I_0(x^*) \) with \( |I| = \kappa \). One can show that \( x^* \) is a local minimum of the cardinality constrained problem if and only if it is a local minimum of all pieces \( \text{NLP}(I) \). But contrary to the cardinality constrained problem the pieces \( \text{NLP}(I) \) are standard nonlinear optimization problems, to which we can apply first and second order necessary and sufficient optimality conditions.

Literature
- O. Burdakov, C. Kanzow, and A. Schwartz: Mathematical Programs with Cardinality Constraints: Reformulation by Complementarity-type Constraints and a Regularization Method
- C. Geiger and C. Kanzow: Theorie und Numerik restringierter Optimierungsaufgaben

Keywords
- cardinality constraints, piecewise decomposition, necessary optimality conditions, sufficient optimality conditions

Goals
- Apply standard first and second order necessary and sufficient optimality conditions to the pieces \( \text{NLP}(I) \) to obtain corresponding optimality conditions for the cardinality constrained problem.
- Compare the resulting optimality conditions to existing ones for cardinality constrained problems.

Prerequisites and Key Aspects

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<td>Nonlinear Optimization</td>
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